THE GYRATOR, AN ELECTRIC NETWORK ELEMENT

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There is a traditional antithesis between the practical mind and the mind inclined to pure scientific pursuits: the former is mainly interested in knowledge that can be put immediate to use, while the latter ponders fundamentals and tries to introduce greater clarity and generality by rigorous considerations which the practical man probably finds superfluous. The invention of the gyrator by Prof. Tellegen is a striking illustration of the service that can be rendered to engineering by the pure scientific approach: originally regarded as a hypothetical possibility, a possibility that had to be recognized for the sake of completeness, the gyrator has subsequently and perhaps rather surprisingly become a reality in the world of microwaves.

Wide use is made in electrical engineering of networks composed of resistors, coils and capacitors, the latter being known as network elements. In use, there is conversion and exchange of energy within and between these network elements. In resistors electrical energy can be converted into heat; in coils and capacitors energy can be stored and later released. The networks are provided with terminal pairs allowing an exchange of energy to take place with the exterior. A terminal pair consists of two conductors between which a voltage of instantaneous value v can exist and through which a current of instantaneous value i can flow (see *fig. 1*). The energy

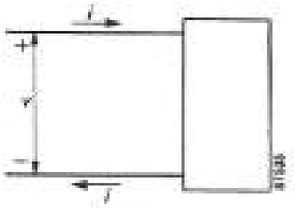

Fig. 1. Network with one terminal pair.

supplied to the network via the terminal pair in a time $\mathrm{d}t$ is $iv\mathrm{d}t$; it can be positive or negative. The network determines relations between the voltages and currents at the terminals, the number of relations being equal to the number of terminal pairs. If for example the network consists of one resistor R, it has one terminal pair and the relation is $v = Ri$.

Networks made up of resistors, coils and capacitors

For the user networks are characterized primarily by the relations that hold between the terminal voltages and currents. As long as the relations between these voltages and currents are the ones desired, the actual composition of the network is, for the user, of secondary concern. It is therefore of great importance to know what sets of such relations are possible with networks consisting of resistors, coils and capacitors. All the possible sets of relations form an arsenal from which the user can take his choice.

In order to build up the whole arsenal, i.e. a complete list of the possible sets of relations, let us first look for general properties of the relations; we can then try to demonstrate that, for each set of relations having these properties, it is possible to design a network of resistors, coils and capacitors for which the said relations are valid. This can be put in another way by saying that we shall try to find necessary and sufficient conditions for the sets of relations if they are to be realizable by networks composed of resistors, coils and capacitors.

General properties of the relations are that they consist of *linear* differential equations relating the terminal voltages and currents, and that the coefficients of the equations are *constant*, i.e. not dependent on time but determined only by the magnitude of the elements composing the networks. Furthermore, the networks contain no source of energy; they are said to be *passive*. Certain properties of the coefficients of the differential equations can be deduced from the fact that the networks are passive. Let us consider, for example, a network with one terminal pair, the properties of which are governed by a differential equation of the first order, so that we can write:

$$a\frac{\mathrm{d}i}{\mathrm{d}t} + bi = c\frac{\mathrm{d}v}{\mathrm{d}t} + dv, \quad . \quad . \quad . \quad (1)$$

where a, b, c and d are constants. From the passivity of the network it can be shown that a, b, c and d all have the same sign.

This can be demonstrated as follows. If we short-circuit the network, in other words if we keep $v = 0$, the current will be determined by:

$$a\frac{\mathrm{d}i}{\mathrm{d}t} + bi = 0.$$

The solution of this differential equation is:

$$i = C\, e^{-\frac{b}{a}t}$$

where C is the constant of integration. In consequence of the passivity, the current in the short-circuited network cannot rise indefinitely and hence a and b have the same sign. It follows in a similar way, by considering the network to be open-circuited, i.e. by keeping $i = 0$, that c and d have the same sign.

For direct current and voltage equation (1) simplifies to

$$bi = dv,$$

so that b/d represents the DC resistance of the network. This must be positive on account of the network's passivity, and hence b and d also have the same sign.

For convenience we shall group the three properties together and say that the sets of relations characterizing networks made up of resistors, coils and capacitors are linear, constant, passive. As far as networks with one terminal pair are concerned, these properties are not only necessary but also sufficient, for Brune has demonstrated [1]) that *any* linear constant passive relation can be realized by a network made up of resistors, coils and capacitors. In particular, it can be demonstrated that any relation having the form of equation (1) can be realized by a network consisting either of two resistors and a coil (if $a/c > b/d$), or of two resistors and a capacitor (if $a/c < b/d$). The two networks are shown in *fig. 2*.

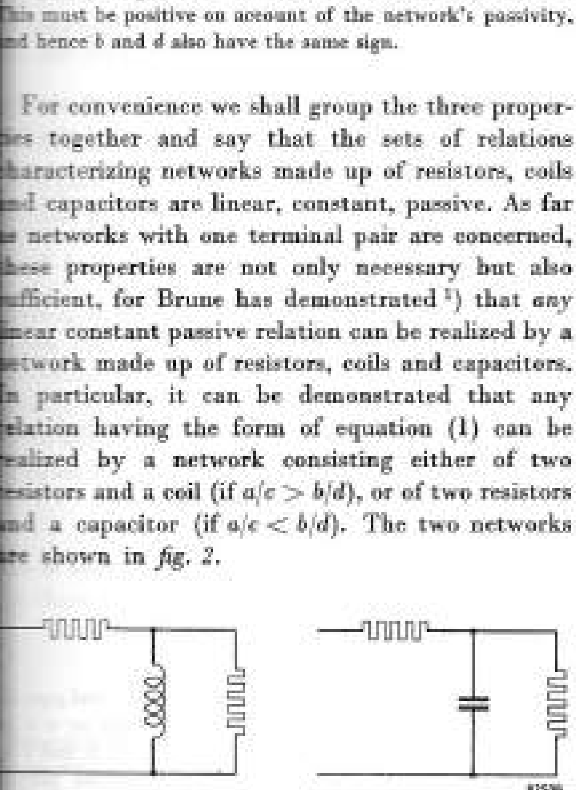

Fig. 2. Networks with one terminal pair and of the first order.

For any network consisting of resistors, coils and capacitors and having more than one terminal pair a set of relations holds that, besides being linear constant passive, has the property known as reciprocity. To explain this property we shall take a network with two terminal pairs and express the two terminal voltages in terms of the two terminal currents; for the present purpose it will be convenient to write the relationships in complex form, rather than use the instantaneous values. Accordingly, we obtain:

$$\left.\begin{aligned} V_1 &= Z_{11}I_1 + Z_{12}I_2, \\ V_2 &= Z_{21}I_1 + Z_{22}I_2, \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

where I_1, I_2, V_1 and V_2 represent the complex values of the terminal currents and voltages. If we assume the sign convention for currents and voltages as indicated in *fig. 3*, it can be shown that always

$$Z_{21} = Z_{12}. \quad \ldots\ldots\ldots \quad (3)$$

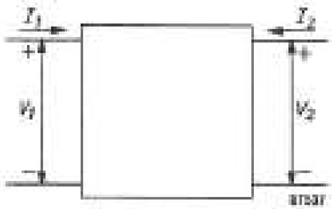

Fig. 3. Network with two terminal pairs.

Expressing the current I_1 through the first terminal pair and the voltage V_2 across the second terminal pair in terms of the two other terminal quantities, we find from equations (2):

$$\left.\begin{aligned} I_1 &= \frac{1}{Z_{11}}V_1 - \frac{Z_{12}}{Z_{11}}I_2, \\ V_2 &= \frac{Z_{21}}{Z_{11}}V_1 + \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{11}}I_2. \end{aligned}\right\} \quad \ldots \quad (4)$$

The coefficient of V_1 in the second equation above is now equal and opposite in sign to the coefficient of I_2 in the first equation. The relation (3) between two coefficients of equations (2) and the relation just discussed between two coefficients of equations (4) are known as the reciprocity relations.

For networks with more than one terminal pair Bayard and others have demonstrated [2]) that any linear constant passive set of relations having the reciprocity property can be realized by a network consisting of resistors, coils and capacitors. Hence the said properties of sets of relations are sufficient as well as necessary.

Our arsenal of all possible sets of relations capable of realization by networks made up of resistors, coils and capacitors is thus complete.

Linear constant passive systems

The result obtained above is not entirely satisfactory, however. We have based our considerations on the resistors, coils and capacitors that had their origin in the laboratory. Electrical engineering has simply accepted these elements for constructing networks. There is something arbitrary and fortuitous about all this. Must we necessarily use these

[1]) O. Brune, J. Math. Phys. **10**, 191, 1931.

[2]) M. Bayard, Bull. Soc. franç. Elect. **9**, 497, 1949.
Also B. D. H. Tellegen, J. Math. Phys. **32**, 1, 1953, which gives further references to the literature.

elements for the building-up of networks? Should an affirmative answer be found to this question, we would be faced with another one: are other elements conceivable, elements that have not been found in the laboratory?

In order to investigate these questions we must not proceed from the basis of the conventional network elements and the networks composed of them but approach the problem from another direction. We must consider "black boxes", systems with terminal pairs that are characterized only by the sets of relations existing between the terminal voltages and currents without concerning ourselves with what is inside them. In what follows we shall confine ourselves to linear constant passive systems, by which we mean systems characterized by linear constant passive sets of relations. We could of course subject ourselves to less stringent limitations: we could also consider systems containing energy sources (active systems, such as amplifiers), systems characterized by linear equations with variable coefficients (variable systems), or systems characterized by non-linear equations (non-linear systems). However, such systems have properties that depart appreciably from those of the networks considered above, and are of a more complex nature. We shall therefore leave them aside.

Let us now try to find the "simplest" kinds of linear constant passive system. These simplest systems we shall call network elements, *by definition*. In this way we shall try to arive at a set of network elements such that any linear constant passive system can be realized as a network composed of them. We can regard such a set of network elements as a complete set.

In order to give a meaning to the epithet "simplest", we shall have to classify our "black boxes". We shall do so according to the number of terminal pairs, the order of the differential equations characterizing them, and whether or not they are able to dissipate electrical energy, i.e. to transform it into heat. This leads, as a first step, to the investigation of systems with one terminal pair, of zero order and involving no dissipation. For such systems the power supplied must be zero at any instant, i.e. $iv = 0$; therefore either $i = 0$ (open terminal pair) or $v = 0$ (short-circuited terminal pair). This does not produce a network element. In accordance with the threefold classification, our next step will be to investigate three classes of systems, namely (1) systems with one terminal pair, of zero order and involving dissipation, (2) systems with one terminal pair, of the first order and involving no dissipation, and (3) systems with two terminal pairs, of zero order and involving no dissipation.

(1) Systems with one terminal pair, of zero order and *involving dissipation*

These systems are characterized by an equation having the form:

$$v = Ri, \quad R > 0 \; . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The fact that R is positive is a consequence of the passivity of the network, this requiring that iv should be positive.

(2) Systems with one terminal pair, of the *first order* and involving no dissipation

Systems with one terminal pair and of the first order are characterized by an equation of the form of (1). It can be shown that this represents a system involving no dissipation either if $b = 0$ and $c = 0$ or if $a = 0$ and $d = 0$. Hence these systems are of two kinds. The first kind is characterized by an equation of the form:

$$v = L\frac{\mathrm{d}i}{\mathrm{d}t}, \quad L > 0 \; . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the second kind by an equation of the form:

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}, \quad C > 0 \; . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The above may be derived as follows. The system to which (1) is applicable has an impedance of

$$Z = \frac{j\omega a + b}{j\omega c + d} \, .$$

In order that there should be no dissipation, the real part of Z must be zero at all frequencies. It follows that $ac = 0$ and $db = 0$. Since $a = 0$ and $b = 0$ means that $Z = 0$, and $c = 0$ and $d = 0$ means that $Z = \infty$, these solutions may be disregarded. Thus either c and b, or a and d, must be zero, and these two cases produce equations (6) and (7), respectively. Since the passivity of the network requires that a, b, c and d should all have the same sign (see above), it further follows that $L = a/d$ and $C = c/b$ are both positive.

Before examining the third class of simplest systems we shall take a closer look at the three systems, (5), (6), and (7), already found. As stated above, we shall regard them as network elements. The latter, then, are defined by equations, not by the physical means required to realize them. For example, we cannot infer from the equation $v = L\,\mathrm{d}i/\mathrm{d}t$ that it describes a "coil". A superconductor has the same equation when we take the effect of the mass of the conducting electrons into account. The kinetic energy of these electrons then takes the place of the magnetic energy of the coil.

The three network elements, so defined, are "ideal"; they can only be realized approximately. For example, (5) is approached by a resistor with low stray capacitance and low self-inductance (6) by a low-loss coil with a low stray capacitance, and (7) by a low-loss capacitor with a low self-inductance (hence the symbols R, L and C used in the equations).

The way in which the network elements are realized is only of secondary importance to the user; it is the external properties given by (5), (6) and (7), that are of primary interest. This remark is similar to that made earlier regarding networks: the user has little interest in their internal composition.

(3) Systems with *two terminal pairs*, of zero order and involving no dissipation

For these systems the total power supplied via the terminal pairs is zero at any instant, so that

$$i_1 v_1 + i_2 v_2 = 0,$$

where i_1, i_2, v_1 and v_2 represent instantaneous values of the terminal currents and voltages, the sign convention for these quantities being assumed to be in accordance with fig. 3. This condition results in two kinds of systems, namely those characterized by equations of the form:

$$\left.\begin{aligned} i_1 &= -n i_2, \\ v_2 &= n v_1, \end{aligned}\right\} \quad \ldots \ldots \quad (8)$$

and those characterized by equations of the form:

$$\left.\begin{aligned} v_1 &= -s i_2, \\ v_2 &= s i_1, \end{aligned}\right\} \quad \ldots \ldots \quad (9)$$

Equations (8) and (9) may be derived as follows. If, for a two-terminal-pair system of zero order, we express the two terminal voltages in terms of the two terminal currents, we can write the equations as:

$$\left.\begin{aligned} v_1 &= a_{11} i_1 + a_{12} i_2, \\ v_2 &= a_{21} i_1 + a_{22} i_2. \end{aligned}\right\} \quad \ldots \ldots \quad (10)$$

It follows from this that:

$$i_1 v_1 + i_2 v_2 = a_{11} i_1^2 + (a_{12} + a_{21})\, i_1 i_2 + a_{22} i_2^2. \quad \ldots \quad (11)$$

In order that (10) should represent a system involving no dissipation, (11) must be zero for all values of i_1 and i_2; hence $a_{11} = 0$, $a_{12} + a_{21} = 0$, and $a_{22} = 0$, which results in (9).

Starting from the equations that express i_1 and i_2 in terms of v_1 and v_2 we also arrive at (9). However, it is also possible to conceive systems in which v_1 and v_2 cannot be expressed in terms of i_1 and i_2, or vice versa. This is the case for systems that are characterized by a relation between i_1 and i_2 and a relation between v_1 and v_2, making it impossible to choose i_1 and i_2 or v_1 and v_2 as independent variables. In order to investigate these systems we can start from equations expressing i_1 and v_2 in terms of v_1 and i_2, or vice versa. We then arrive at (8).

We shall regard the systems thus found, (8) and (9), as two further network elements. System (8) is called an ideal *transformer* and we have proposed [3] the name ideal *gyrator* for system (9). Observations similar to those made above on (5), (6) and (7) may also be made on these two systems.

An approximation to the ideal transformer is given by two tightly coupled low-loss coils of large self-inductances. This is in fact the way in which the concept first arose. The theoretical route by which we have now arrived at the same concept shows why we should regard the ideal transformer as a separate network element, defined, in fact, by (8). From the remarks on reciprocity in connection with equations (4) we see that the ideal transformer also possesses this property. The ideal gyrator does *not* possess the property of reciprocity, as will be clear from a comparison of (9) with (2) and (3). It is therefore impossible to obtain an approximation to it by combining resistors, coils and capacitors, because such a combination always has the property of reciprocity, as we saw above. For the realization of the gyrator other physical means are necessary. One way of realizing it, making use of gyromagnetic effects in ferromagnetic materials, will be described in a further article in this Review [4]).

The addition of the gyrator to the set of network elements has extended the list of possible sets of relations between terminal voltages and currents. This extension was made possible by the fact that we started from linear constant passive systems as such, without imposing on them the condition of reciprocity. The fact that system (9), while being linear constant passive lacks the property of reciprocity, shows that the latter is not a consequence of linearity, constancy and passivity, and that imposing it does indeed constitute a limitation of the possibilities.

Some properties of the gyrator

The ideal gyrator has the property of "gyrating" a current into a voltage, and vice versa. The coefficient s, which has the dimension of a resistance, we call the gyration resistance; $1/s$ we call the gyration conductance. We shall represent the gyrator in circuit diagrams by the symbol shown in *fig. 4*.

The following properties of the ideal gyrator can be easily derived from (9).

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[3]) B. D. H. Tellegen, Philips Res. Rep. 3, 81, 1948.
[4]) H. G. Beljers, Application of ferroxcube to uni-directional waveguides, to appear shortly in this Review.

If we have the output terminals open-circuited, i.e. $i_2 = 0$, the input terminals are short-circuited,

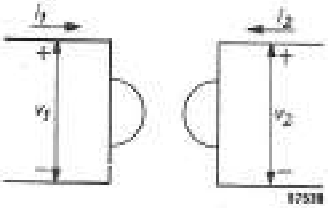

Fig. 4. Symbol for the gyrator.

i.e. $v_1 = 0$, and vice versa. If we connect an inductance L or a capacitance C across the output terminals, we find a capacitance L/s^2 or, in the second case, an inductance s^2C between the input terminals. In general, if we connect an impedance Z across the output terminals, we find an impedance s^2/Z between the input terminals. An impedance Z in series or in parallel with the output terminals has the same effect as an impedance s^2/Z in parallel or in series, respectively, with the input terminals (*fig. 5*).

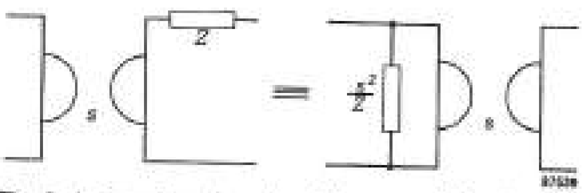

Fig. 5. An impedance in series with one terminal pair of an ideal gyrator is equivalent to another impedance in parallel with the other terminal pair.

Two ideal gyrators in cascade form an ideal transformer; an ideal gyrator and an ideal transformer in cascade form another ideal gyrator.

The combination of a gyrator with a resistor shown in *fig. 6a* gives a system to which the following apply:

$$\left.\begin{aligned} v_1 &= Ri_1 + (R-s)i_2, \\ v_2 &= (R+s)i_1 + Ri_2. \end{aligned}\right\} \quad \ldots \quad (12)$$

Thus an input i_1 gives rise to a voltage component $(R+s)i_1$ across the output; an output current i_2 gives rise to a voltage component $(R-s)i_2$ across the input. If $R = s$, the latter component is zero, so that the input voltage is independent of the output current and dependent only on the input current. In such a case we may say that the system transmits only in the direction from input to output, and not in the reverse direction. The system shown in fig. 6*b* has properties of a similar kind; this can be easily demonstrated by writing down the equations expressing the terminal currents in terms of the terminal voltages.

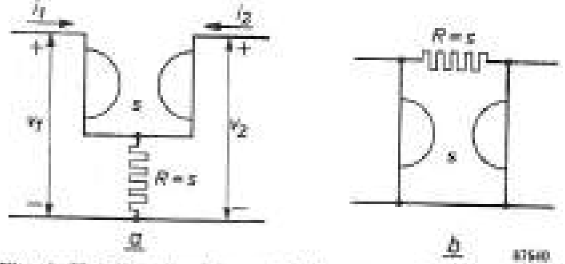

Fig. 6. Uni-directional networks, i.e. networks that transmit only in one direction.

No further elements remain to be added to the five ideal linear constant passive network elements: Oono and Yasuura have demonstrated [5]) that any linear constant passive system can be realized as a network composed of these five network elements. The five ideal network elements therefore form a complete set.

[5]) Y. Oono and K. Yasuura, Mem. Fac. Engng., Kyushu Univ. 14, 124, 1954; also in Ann. Télécomm. 9, pp. 73 and 109, 1954.

Summary. The writer considers the electric networks having one or more terminal pairs that can be built up from conventional network elements, namely resistors, coils and capacitors. The set of relations between terminal voltages and currents determined by such a network has properties that correspond to the linear, constant and passive nature of such networks and, furthermore, has the property of reciprocity. It has been demonstrated that, conversely, these properties are sufficient for any set of relation possessing them to be realizable by a network. The writer reverses this line of reasoning and raises the question: what (ideal) network elements must be introduced in order to make it possible to realize all linear constant passive systems. There are no grounds for including reciprocity amongst the properties imposed. It is then shown that, apart from the conventional elements — resistor, coil, capacitor, and the ideal transformer — a new network element has to be introduced; this new element has been christened ideal gyrator. Networks containing the new element generally lack the property of reciprocity. A brief sketch is given of some of the properties of the gyrator.